MATH5360 Game Theory
Exercise 5

1. (a) For coalition $S_1 = A_1, A_2$, the game bimatrix for the 2-person game between S and S^c is

$$(A_1, B_1) = \begin{pmatrix} (1,5) & (-1,7) \\ (6,0) & (7,-1) \\ (-1,7) & (0,6) \\ (8,-2) & (4,2) \end{pmatrix}.$$

A_1 can be reduced to

$$A_1' \begin{pmatrix} 6 & 7 \\ 8 & 4 \end{pmatrix},$$

thus $\nu(S_1) = \frac{6\times4-8\times7}{6+4-8-7} = 32/5$ and $\nu(A_3) = \nu(S^c) = 6 - 32/5 = -2/5$. Similarly, We have $\nu(A_1, A_3) = 8$ and $\nu(A_2) = 6 - 8 = -2$, $\nu(A_2, A_3) = 15/2$ and $\nu(A_1) = 6 - 15/2 = -3/2$, and $\nu(A_1, A_2, A_3) = 6$.

(b) Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation, then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq -3/2,\ x_2 \geq -2,\ x_3 \geq -2/5, \\ x_1 + x_2 \geq 32/5,\ x_1 + x_3 \geq 8,\ x_2 + x_3 \geq 15/2, \\ x_1 + x_2 + x_3 = 6. \end{cases} \tag{1}$$

$-3/2 \leq x_1 = 6 - x_2 - x_3 \leq 6 - 15/2$ implies $x_1 = -3/2$, $-2 \leq x_2 = 6 - x_1 - x_3 \leq 6 - 8$ implies $x_2 = -2$, $-2/5 \leq x_3 = 6 - x_1 - x_1 \leq 6 - 32/5$ implies $x_3 = -2/5$, but $x_1 + x_2 + x_3 = -39/10 \neq 6$. Thus, $C(\nu) = \emptyset$.

2. Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation, then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq 27,\ x_2 \geq 8,\ x_3 \geq 18, \\ x_1 + x_2 \geq 36,\ x_1 + x_3 \geq 50,\ x_2 + x_3 \geq 27, \\ x_1 + x_2 + x_3 = 60. \end{cases} \tag{2}$$

which is equivalent to

$$\begin{cases} 33 \geq x_1 \geq 27, \\ 10 \geq x_2 \geq 8, \\ 42 \geq x_1 + x_2 = 60 - x_3 \geq 36. \end{cases} \tag{3}$$

$C(\nu)$ is the intersecting region of the three strip regions in Figure 1.

3. (a) $\mu(1) = \mu(2) = \mu(3) = 0$ and $\mu(1,2,3) = 1$.

$$k = \frac{1}{\nu(1,2,3) - \nu(1) - \nu(2) - \nu(3)} = 1/7,$$

and we have

$$\mu(1,2) = k(\nu(1,2) - \nu(1) - \nu(2)) = 2/7,$$
$$\mu(1,3) = k(\nu(1,3) - \nu(1) - \nu(3)) = 3/7,$$

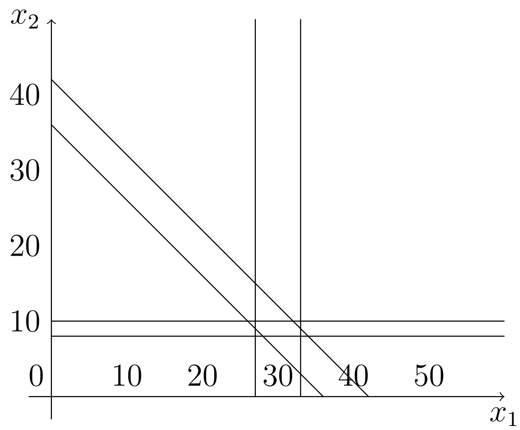

Figure 1:

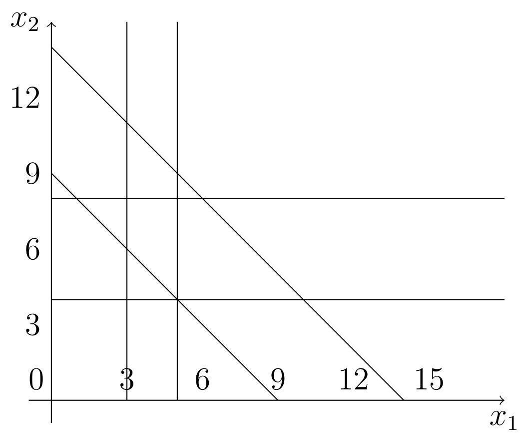

Figure 2:

$$\mu(2,3) = k(\nu(2,3) - \nu(2) - \nu(3)) = 5/7.$$

(b) Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation, then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq 3, \ \ x_2 \geq 4, \ \ x_3 \geq 6, \\ x_1 + x_2 \geq 9, \ \ x_1 + x_3 \geq 12, \ \ x_2 + x_3 \geq 15, \\ x_1 + x_2 + x_3 = 20. \end{cases} \tag{4}$$

which is equivalent to

$$\begin{cases} 5 \geq x_1 \geq 3, \\ 8 \geq x_2 \geq 4, \\ 14 \geq x_1 + x_2 = 20 - x_3 \geq 9. \end{cases} \tag{5}$$

$C(\nu)$ is the intersecting region of the three strip regions in Figure 2.

(c) $\phi_1 = 9/2$, $\phi_2 = 13/2$ and $\phi_3 = 9$.

4. (a) $\nu(A) = \nu(B) = \nu(C) = 0$, $\nu(A, B) = 3$, $\nu(A, C) = 5$, $\nu(B, C) = 2$ and $\nu(A, B, C) = 6$.

(b) $\phi_1 = 8/3$, $\phi_2 = 7/6$ and $\phi_3 = 13/6$.

(c) According to the Shapley's value, A,B,C should pay 25/3, 35/6, 35/6 respectively.

5. We have $\nu(1) = \nu(2) = \nu(3) = \nu(4) = 0$, $\nu(1,2) = \nu(1,3) = \nu(1,4) = 1$, $\nu(2,3) = \nu(2,4) = \nu(3,4) = 0$ and $\nu(S) = 1$ for any S with $|S| \geq 3$. Thus, $\phi_1 = 1/2$, $\phi_2 = \phi_3 = \phi_4 = 1/6$.

9. (a) $\nu(\{A\}) = \nu(\{B\}) = \nu(\{C\}) = 0$, $\nu(\{A, B\}) = 72$, $\nu(\{A, C\}) = 40$, $\nu(\{B, C\}) = 40$ and $\nu(\{A, B, C\}) = 176$.

(b) $\mu(\{A\}) = \mu(\{B\}) = \mu(\{C\}) = 0$ and $\mu(\{A, B, C\}) = 1$.

$$k = \frac{1}{\nu(\{A, B, C\}) - \nu(\{A\}) - \nu(\{B\}) - \nu(\{C\})} = 1/176,$$

and we have

$$\mu(\{A, B\}) = k(\nu(\{A, B\}) - \nu(\{A\}) - \nu(\{B\})) = 9/22.$$

(c) Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation, then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \\ x_1 + x_2 \geq 72, \; x_1 + x_3 \geq 40, \; x_2 + x_3 \geq 40, \\ x_1 + x_2 + x_3 = 176. \end{cases} \tag{6}$$

which is equivalent to

$$\begin{cases} 136 \geq x_1 \geq 0, \\ 136 \geq x_2 \geq 0, \\ 176 \geq x_1 + x_2 = 176 - x_3 \geq 72. \end{cases} \tag{7}$$

$C(\nu)$ is the intersecting region of the three strip regions in Figure 3.

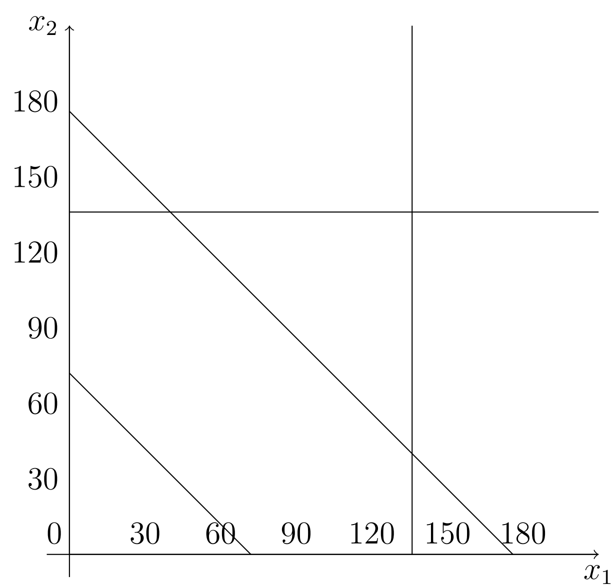

Figure 3: